A GENERALIZATION OF MULTIPLIER RULES FOR INFINITE-DIMENSIONAL OPTIMIZATION PROBLEMS

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Abstract. We provide a generalization of first-order necessary conditions of optimality for infinite-dimensional optimization problems with a finite number of inequality constraints and with a finite number of inequality and equality constraints. Our assumptions on the differentiability of the functions are weaker than those of existing results.

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1. Introduction

We provide an improvement of first-order necessary conditions of optimality for infinite-dimensional problems under a finite number of inequality constraints and under a finite number of inequality and equality constraints in the form of Fritz John's theorem and Karush-Kuhn-Tucker's theorem.
In this paper, we give a proof of multiplier rules by following the same approach as Michel in [1] p. 510. To prove his result, Michel uses the Brouwer fixed-point theorem that is why, at the first slight, his proof seems specific to finite-dimensional optimisation problems. However, we remark that we can extend this result for infinite-dimensional optimizaton problems. The proof of Michel is explained in detail in [2], Appendix B. Another proof of the multiplier rules was established by Halkin in [3] but his proof is completely different. Indeed, Halkin uses an implicit function theorem with only Fréchet differentiable at a point framework instead of the continuously Fréchet differentiable framework. The improvement of Michel and Halkin is to replace the assumption of continuously Fréchet differentiable on a neighborhood of the optimal solution (see in [4] Chapter 13 section 2) with the assumptions of the continuity on a neighborhood of the optimal solution and the Fréchet differentiability at the optimal solution.
Note that there are another way to generalize the assumption of continuous Fréchet differentiability by using locally Lipschitzian mappings e.g. [5]. The statement of Halkin and Michel is not similar with the statements of locally Lispchitzian. Indeed, in general, a mapping which is Fréchet differentiable at a point is not locally Lipschitzian arround this point and conversely a mapping which is locally Lipschitzian arround a point is not Fréchet differentiable at this point.
In [6], Blot gave also a proof of multiplier rules for finite-dimensional optimization problems under only inequality constraints and under inequality and equality constraints. For the problems with inequality constraints, Blot reduced the assumptions of Pourciau in [7] by replacing, at the optimal solution, the Fréchet

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differentiability with the Gâteaux differentiability. For the problems with inequality and equality constraints, Blot deleted the assumptions of local continuity on a neighborhood for the objective function and for the functions in the constraints of inequality. Therefore, Blot improved the multilplier rules of Michel and Halkin by lightening the assumptions on the continuity of the functions.
The main contributions of this paper are as follows.

- Contrary to Blot and Michel, we do not assume the finiteness of the dimension of the space. Therefore, we extend the main theorems of [6] and [1] in infinite-dimensional vector spaces.
- Moreover, in comparison with Blot's multiplier rules, we replaced the assumptions of Fréchet differentiability by the Hadamard differentiability which is weaker in infinite-dimensional vector. For consequently, our assumptions on the differentiability of the functions are weaker than [6], [1] and [3].

We summarize the content of this paper as follows.
In Section 2, we state the main theorems of the paper.
In Section 3, we specify the definition of Gâteaux differentiability and Hadamard differentiability. Besides, we recall a supporting hyperplane theorem and the Schauder fixed-point theorem.
In Section 4, in order to proof our first-order necessary conditions under inequality constraints, we delete the inactive inequality constraints. Next, we use a supporting hyperplane theorem to find the multipliers.
In Section 5, we give a proof of first-order necessary conditions of optimality under inequality and equality constraints. As in Section 4, we delete the inactive inequality constraints. In order to use the supporting hyperplane theorem, we use the Schauder fixed-point theorem.

2. Statements of the Main Results

The paper deals with infinite-dimensional optimization problems with a finite list of inequality constraints and with a finite list of inequality and equality constraints. Let E be a normed vector space, let Ω be a nonempty open subset of E, let $f_i : \Omega \to \mathbb{R}$ when $i \in \{0, ..., m\}$ be functions, let $f : \Omega \to \mathbb{R}$, $g_i : \Omega \to \mathbb{R}$ when $i \in \{1, ..., p\}$, $h_j : \Omega \to \mathbb{R}$ when $j \in \{1, ..., q\}$ be functions and m, p and q are integer number. We consider the two following problems

$$(\mathcal{I}) \begin{cases} \text{Maximize} & f_0(x) \\ \text{subject to} & x \in \Omega \\ & \forall i \in \{1, ..., m\},\ f_i(x) \geq 0 \end{cases}$$

and

$$(\mathcal{P}) \begin{cases} \text{Maximize} & f(x) \\ \text{subject to} & x \in \Omega \\ & \forall i \in \{1, ..., p\},\ g_i(x) \geq 0 \\ & \forall j \in \{1, ..., q\},\ h_j(x) = 0. \end{cases}$$

The main theorems of the paper are the following ones.

Theorem 2.1. *Let $\hat{x}$ be a solution of $(\mathcal{I})$. We assume that the following assumptions are fulfilled.*
(i) *For all $i \in \{0, ..., m\}$, f_i is Gâteaux differentiable at $\hat{x}$.*
(ii) *For all $i \in \{1, ..., m\}$, f_i is lower semicontinuous at $\hat{x}$ when $f_i(\hat{x}) > 0$.*

Then, there exist $\lambda_0, ..., \lambda_m \in \mathbb{R}_+$ *which satisfy the following conditions.*
(a) $(\lambda_0, ..., \lambda_m) \neq (0, ..., 0)$.
(b) For all $i \in \{1, ..., m\}$, $\lambda_i f_i(\hat{x}) = 0$.
(c) $\sum_{i=0}^{m} \lambda_i D_G f_i(\hat{x}) = 0$.
In addition, if we assume that the following assumption is verified
(iii) there exists $w \in E$ such that for all $i \in \{1, ..., m\}$, $D_G f_i(\hat{x})w > 0$ when $f_i(\hat{x}) = 0$ then we can take
(d) $\lambda_0 = 1$.

Theorem 2.2. *Let $\hat{x}$ be a solution of $(\mathcal{P})$. We assume that the following assumptions are fulfilled.*
(i) *f is Hadamard differentiable at $\hat{x}$.*
(ii) *For all $i \in \{1, ..., p\}$, g_i is Hadamard differentiable at $\hat{x}$ when $g_i(\hat{x}) = 0$.*
(iii) *For all $i \in \{1, ..., p\}$, g_i is lower semicontinuous at $\hat{x}$ and Gâteaux differentiable at $\hat{x}$ when $g_i(\hat{x}) > 0$.*
(iv) *For all $j \in \{1, ..., q\}$, h_j is continuous on a neighborhood at $\hat{x}$ and Hadamard differentiable at $\hat{x}$.*
Then, there exist $\lambda_0, ..., \lambda_p \in \mathbb{R}_+$ and $\mu_1, ..., \mu_q \in \mathbb{R}$ which satisfy the following conditions.
(a) $(\lambda_0, ..., \lambda_p, \mu_1, ..., \mu_q) \neq (0, ..., 0)$.
(b) For all $i \in \{1, ..., p\}$, $\lambda_i g_i(\hat{x}) = 0$.
(c) $\lambda_0 D_H f(\hat{x}) + \sum_{i=1}^{p} \lambda_i D_G g_i(\hat{x}) + \sum_{j=1}^{q} \mu_j D_H h_j(\hat{x}) = 0$.
Futhermore, if we assume that the following assertion hold
(v) $D_H h_1(\hat{x}), ..., D_H h_q(\hat{x})$ are linearly independent
we have
(d) $(\lambda_0, ..., \lambda_p) \neq (0, ..., 0)$.
Moreover, under (v) and the following assertion
(vi) there exists $w \in \cap_{i=1}^{q} Ker D_H hj(\hat{x})$ such that for all $i \in \{1, ..., p\}$, $D_G g_i(\hat{x})w > 0$ when $g_i(\hat{x}) = 0$
we can take
(e) $\lambda_0 = 1$.

3. Recall and Notations

We set $\mathbb{N}$ the set of positive integer and $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$. $\mathbb{R}$ denotes the set of real numbers and $\mathbb{R}_+$ the set of non negative real numbers.
Let E, F and G be three normed vector spaces, let Ω be a nonempty open subset of E, let $f : \Omega \to F$ be a mapping, let $x \in \Omega$, let $y \in E$ and $r \in]0, +\infty[$. The closed ball centered at y with a radius equal to r is denoted by $\overline{B}(y, r)$.
Let $A \subset E$ and $B \subset F$, $C^0(A, B)$ denotes the continuous mappings from A into B. bd$A$ denotes the topological boundary of A.
We denote by $\mathcal{L}(E, F)$ the space of the bounded linear mappings from E into F.
Let $l \in \mathcal{L}(E, F)$, we note $Iml = l(E)$. Let $l_1 \in \mathcal{L}(E, F)$ and $l_2 \in \mathcal{L}(E, G)$, we note by (l_1, l_2) the mapping in $\mathcal{L}(E, F \times G)$ defined by for all $x \in E$, $(l_1, l_2)x = (l_1 x, l_2 x)$.
f is called Gâteaux differentiable at x when there exists $D_G f(x) \in \mathcal{L}(X, Y)$ such that for all $h \in E$, $\lim_{t \downarrow 0} \frac{f(x+th) - f(x)}{t} = D_G f(x)h$.
We say that f is Hadamard differentiable at x when there exists $D_H f(x) \in \mathcal{L}(X, Y)$ such that for all $h \in E$, for all sequence $(h_n)_{n \in \mathbb{N}}$ converging to h and for all sequence $(t_n)_{n \in \mathbb{N}}$ of positive numbers converging to 0 we have

$\lim_{n\to+\infty} \frac{f(x+t_nh_n)-f(x)}{t_n} = D_Hf(x)h$ which is equivalent to (see [8] p. 265) for each K compact in E, $\lim_{t\downarrow 0}\sup_{h\in K} \frac{f(x+th)-f(x)}{t} = D_Hf(x)h$.
When f is Hadamard differentiable at x, f is also Gâteaux differentiable at x and $D_Hf(x) = D_Gf(x)$. But the converse is false when the dimension of E is greater than 2.
More information on these notions can be found in [8].
If $n \in \mathbb{N}^*$, we note $\langle\cdot,\cdot\rangle$ the canonical scalar product on $\mathbb{R}^n$, $(e_{n,i})_{1\leq i\leq n}$ the canonical basis of $\mathbb{R}^n$ and $\|\cdot\|_\infty$ the maximum norm on $\mathbb{R}^n$. Moreover, we note by $\overline{B}_{\|\cdot\|_\infty}(y,r)$ the closed ball centered at y with a radius equal to r in $\mathbb{R}^n$ with the maximum norm.
We recall a supporting hyperplane theorem.

Theorem 3.1. *Let $n \in \mathbb{N}^*$. Let C be a nonempty convex subset of $\mathbb{R}^n$ and $z \in \mathrm{bd}C$. Then there exist $v \in \mathbb{R}^n \setminus \{0\}$ and $\gamma \in \mathbb{R}$ such that $\langle v, z\rangle = \gamma$ and for all $x \in C$, $\langle v, x\rangle \leq \gamma$.*

This theorem is a corralary of Hahn-Banach theorem. We can find a proof in [9] p. 37. Note that if $z = 0$ we have $\gamma = 0$.
Next, we recall the Schauder fixed-point theorem.

Theorem 3.2. *(Schauder fixed-point theorem) Let E be a normed vector space, let C be a nonempty convex and compact subset of E and let $f : C \to C$ be a continuous mapping, then f admit a fixed point i.e. there exists $x \in C$ such that $f(x) = x$.*

We can find a proof of the Schauder fixed-point theorem in [10] p. 119.

4. Proof of Theorem 2.1

We set $S := \{i \in \{1, ..., m\} : \ f_i(\hat{x}) = 0\}$. If $S = \emptyset$ we have for all $i \in \{1, ..., m\}$, $f_i(\hat{x}) > 0$ using the lower semicontinuous of f_i, there exists an open neighborhood Ω_1 of $\hat{x}$ in Ω such that for all $i \in \{1, ..., m\}$, for all $x \in \Omega_1$, $f_i(x) > 0$. Since $\hat{x}$ is a solution of $(\mathcal{I})$, we have $\hat{x}$ maximize f_0 on Ω_1. Therefore by using (i), we have $D_Gf_0(\hat{x}) = 0$. By taking $\lambda_0 = 1$ and for all $i \in \{1, ..., m\}$, $\lambda_i = 0$, we proved (a), (b), (c) and (d). We assume that $S \neq \emptyset$ in the rest of the proof.

4.1. **To delete all inactive inequality constraints.** By doing a change of index, we can assume that $S = \{1, ..., s\}$ where $1 \leq s \leq m$. Since for all $i \in \{s+1, ..., m\}$, we have $f_i(\hat{x}) > 0$, using (ii) there exists an open neighborhood U of $\hat{x}$ in Ω such that for all $i \in \{s+1, ..., m\}$, for all $x \in U$, we have $f_i(x) > 0$. For consequently, we have $\hat{x}$ is a solution of the following problem

$$(\mathcal{N}I)\begin{cases} \text{Maximize} & f_0(x) \\ \text{subject to} & x \in U \\ & \forall i \in \{1, ..., s\},\ f_i(x) \geq 0. \end{cases}$$

4.2. **Proof of (a), (b), (c).** We consider the mapping $F : U \to \mathbb{R}^{s+1}$ defined by $\forall x \in U$, $F(x) = (f_0(x), ..., f_s(x))$. Since for all $i \in \{0, ..., s\}$, f_i is Gâteaux differentiable at $\hat{x}$, we have F is Gâteaux differentiable at $\hat{x}$ and
$D_GF(\hat{x}) = (D_Gf_0(\hat{x}), ..., D_Gf_s(\hat{x}))$.
We set $C := ImD_GF(\hat{x}) + \mathbb{R}^{s+1}_-$. We note that C is a convex set of $\mathbb{R}^{s+1}$. Moreover, C is not a neighborhood of 0. To prove this, we proceed by contradiction, by assuming that C is a neighborhood of 0. Therefore, there exists $r > 0$ such that

$\overline{B}_{\|\cdot\|_\infty}(0,r) \subset C$. Since $b = (r, ..., r) \in \overline{B}_{\|\cdot\|_\infty}(0,r)$ we have $b \in C$ then there exists $u \in E$ and $z = (z_0, ..., z_s) \in \mathbb{R}_-^{s+1}$ such that $D_G F(\hat{x})u + z = b$. For consequently, we have

$$\forall i \in \{0, ..., s\}, D_G f_i(\hat{x}).u = r - z_i \geq r. \tag{4.1}$$

By using (4.1), we remark that $u \neq 0$.
Since F is Gâteaux differentiable at $\hat{x}$, we have

$$\exists \delta > 0\ \forall t \in]0, \delta]\ (\hat{x} + tu \in U)\ \|F(\hat{x} + tu) - F(\hat{x}) - tD_G F(\hat{x})u\|_\infty < rt. \tag{4.2}$$

Then, using (4.2) with $t = \delta$, we have $\|F(\hat{x}+\delta u) - F(\hat{x}) - \delta D_G F(\hat{x})u\|_\infty < r\delta$ which implies that $\forall i \in \{0, ..., s\}, f_i(\hat{x}+\delta u) - f_i(\hat{x}) - \delta D_G f_i(\hat{x})u > -r\delta$. For consequently, by using (4.1) we have for all $i \in \{0, ..., s\}, f_i(\hat{x}+\delta u) - f_i(\hat{x}) > \delta D_G f_i(\hat{x})u - \delta r \geq 0$. Therefore, we have $f_0(\hat{x} + \delta u) > f_0(\hat{x})$ and for all $i \in \{0, ..., s\},\ f_i(\hat{x} + \delta u) > 0$ which implies that $\hat{x}$ is not a solution of $(\mathcal{N}I)$. This is a contradiction. Since $0 \in C$ and C is not a neighborhood of 0, we have $0 \in \mathrm{bd}C$.
Since C is a convex of $\mathbb{R}^{s+1}$ and $0 \in \mathrm{bd}C$, by using Theorem 3.1 there exists $v = (\lambda_0, ..., \lambda_s) \in \mathbb{R}^{s+1} \setminus \{0\}$ such that for all $x \in C$, $\langle v, x\rangle \leq 0$. For consequently, we have

$$\forall u \in E,\ \forall z = (z_0, ..., z_s) \in \mathbb{R}_-^{s+1},\ \textstyle\sum_{i=0}^s \lambda_i (D_G f_i(\hat{x})u + z_i) \leq 0. \tag{4.3}$$

We set for all $i \in \{s + 1, ..., m\}$ $\lambda_i = 0$.
Since $(\lambda_0, ..., \lambda_s) \neq 0$, we have $(\lambda_0, ..., \lambda_m) \neq 0$. Let $i \in \{0, ..., s\}$, by using (4.3) with $u = 0$ and $z = -e_{s+1,i+1}$, we have $-\lambda_i \leq 0$ which implies that $\lambda_i \geq 0$.
We have also for all $i \in \{1, ..., m\},\ \lambda_i f_i(\hat{x}) = 0$.
By using (4.3), with $z = 0$, we have $\forall u \in E,\ \sum_{i=0}^s \lambda_i D_G f_i(\hat{x})u \leq 0$ which implies that

$$\textstyle\sum_{i=0}^s \lambda_i D_G f_i(\hat{x}) = 0, \tag{4.4}$$

therefore $\sum_{i=0}^m \lambda_i D_G f_i(\hat{x}) = 0$.
We proved (a), (b) and (c).

4.3. **Proof of (d).** In addition, if we assume (iv), we have $\lambda_0 \neq 0$. We proceed by contradiction by assuming that $\lambda_0 = 0$. Since (iii) and $(\lambda_1, ..., \lambda_s) \in \mathbb{R}_+^s \setminus \{0\}$, we have $\sum_{i=1}^s \lambda_i D_G f_i(\hat{x})w > 0$. By using (4.4), we have $\sum_{i=1}^s \lambda_i D_G f_i(\hat{x})w = 0$. This a contradiction. Since, $\lambda_0 \neq 0$, by taking for all $i \in \{0, ..., m\}$, $\lambda'_i = \frac{\lambda_i}{\lambda_0}$, we proved (d).

5. Proof of Theorem 2.2

We set $S := \{i \in \{1, ..., p\} :\ g_i(\hat{x}) = 0\}$. Without loss of generality, we can assume that $S \neq \emptyset$. If $S = \emptyset$ we can delete all inequality constraints. Indeed, we have for all $i \in \{1, ..., p\}$, $g_i(\hat{x}) > 0$, by using (iii), there exists a neighborhood Ω_1 of $\hat{x}$ in Ω such that for all $i \in \{1, ..., p\}$, for all $x \in \Omega_1$, $g_i(x) > 0$. For consequently, $\hat{x}$ is a solution of the following problem

$$(\mathcal{S}P) \begin{cases} \text{Maximize} & f(x) \\ \text{subject to} & x \in \Omega_1 \\ & \forall j \in \{1, ..., q\},\ h_j(x) = 0. \end{cases}$$

5.1. **To delete all inactive inequality constraints.** In the rest of the proof, we assume that $S \neq \emptyset$. By doing a change of index, we can assume that $S = \{1, ..., s\}$ where $1 \leq s \leq m$. Since for all $i \in \{s+1, ..., m\}$, we have $g_i(\hat{x}) > 0$, using (iii) and (iv) there exists an open neighborhood U of $\hat{x}$ in Ω such that for all $i \in \{s+1, ..., m\}$, for all $x \in U$, we have $g_i(x) > 0$ and for all $j \in \{1, ..., q\}$, h_j is continuous on U. For consequently, we have $\hat{x}$ is a solution of the following problem

$$(\mathcal{N}P) \begin{cases} \text{Maximize} & f(x) \\ \text{subject to} & x \in U \\ & \forall i \in \{1, ..., s\},\ g_i(x) \geq 0 \\ & \forall j \in \{1, ..., q\},\ h_j(x) = 0. \end{cases}$$

5.2. **Proof of (a), (b), (c).** We consider the mappings $G : U \to \mathbb{R}^{s+1}$ and $H : U \to \mathbb{R}^q$ defined by $\forall x \in U$, $G(x) = (f(x), g_1(x), ..., g_s(x))$ and $H(x) = (h_1(x), ..., h_q(x))$. Since (i), (ii) and (iv) we have G and H are Hadamard differentiable at $\hat{x}$. Moreover $D_H G(\hat{x}) = (D_H f(\hat{x}), D_H g_1(\hat{x}) ..., D_H g_s(\hat{x}))$ and $D_H H(\hat{x}) = (D_H h_1(\hat{x}), ..., D_H h_q(\hat{x}))$.

We set $C := Im(D_H G(\hat{x}), D_H H(\hat{x})) + \mathbb{R}_-^{s+1} \times \{0\}$. C is a convex set of $\mathbb{R}^{s+q+1}$.

C is not a neighborhood of 0. To prove this, we proceed by contradiction, by assuming that C is a neighborhood of 0. Therefore, there exists $r > 0$ such that $\overline{B}_{\|\cdot\|_\infty}(0, r) \subset C$.

We set $b = (r, ..., r) \in \mathbb{R}^{s+1}$. Since, for all $j \in \{1, ..., q\}$, $(b, re_{q,j}) \in C$ and $(b, -re_{q,j}) \in C$, there exists $u_j \in E$ and $z_j = (z_{0,j}, ..., z_{s,j}) \in \mathbb{R}_-^{s+1}$ such that

$$D_H G(\hat{x}) u_j + z_j = b \text{ and } D_H H(\hat{x}) u_j = re_{q,j} \tag{5.1}$$

and there exists $\tilde{u}_j \in E$ and $\tilde{z}_j = (\tilde{z}_{0,j}, ..., \tilde{z}_{s,j}) \in \mathbb{R}_-^{s+1}$ such that

$$D_H G(\hat{x}) \tilde{u}_j + \tilde{z}_j = b \text{ and } D_H H(\hat{x}) \tilde{u}_j = -re_{q,j}. \tag{5.2}$$

We set $K := \{\sum_{j=1}^q a_j u_j + \sum_{j=1}^q \tilde{a}_j \tilde{u}_j : \forall j \in \{1, ..., q\},\ a_j \geq 0, \tilde{a}_j \geq 0$ and $\sum_{j=1}^q a_j + \sum_{j=1}^q \tilde{a}_j = 1\}$. By using (5.1), we have $0 \notin K$.

We remark that K is a convex and compact set of E. Since G and H are Hadamard differentiable at $\hat{x}$ and K is a compact set of E, we have

$$\exists \delta_1 > 0\ \forall t \in]0, \delta_1],\ \forall k \in K,\ \|G(\hat{x} + tk) - G(\hat{x}) - tD_H G(\hat{x})k\|_\infty < rt \tag{5.3}$$

$$\exists \delta_2 > 0\ \forall t \in]0, \delta_2],\ \forall k \in K,\ \|H(\hat{x} + tk) - H(\hat{x}) - tD_H H(\hat{x})k\|_\infty < \tfrac{r}{q} t. \tag{5.4}$$

Since U is a neighborhood of $\hat{x}$, we have there exists $r_0 > 0$ such that $\overline{B}(\hat{x}, r_0) \subset U$. we set $\alpha := \min\{\delta_1, \delta_2, \frac{r_0}{\sup_{k \in K} \|k\|}\}$. Therefore, by using (5.3) and (5.4) with $t = \alpha$, we have

$$\forall k \in K,\ |f(\hat{x} + \alpha k) - f(\hat{x}) - \alpha D_H f(\hat{x})k| < r\alpha \tag{5.5}$$

$$\forall k \in K, \forall i \in \{1, ..., s\},\ |g_i(\hat{x} + \alpha k) - \alpha D_H g_i(\hat{x})k| < r\alpha \tag{5.6}$$

$$\forall k \in K, \forall j \in \{1, ..., q\},\ |h_j(\hat{x} + \alpha k) - \alpha D_H h_j(\hat{x})k| < \tfrac{r}{q}\alpha. \tag{5.7}$$

We set for all $j \in \{1, ..., q\}$, for all $k \in K$, $w_j(k) = \frac{1}{\alpha} h_j(\hat{x} + \alpha k) - D_H h_j(\hat{x})k$. We note that for all $j \in \{1, ..., q\}$, $w_j \in C^0(K, \mathbb{R})$ because $h_j \in C^0(K, \mathbb{R})$ and for all $j \in \{1, ..., q\}$, for all $k \in K$, $|w_j(k)| < \frac{r}{q}$ (by (5.7)).

We consider the mapping $\Phi : K \to E$ defined by $\Phi(k) := \sum_{j=1}^q \phi_j(k) u_j + \sum_{j=1}^q \tilde{\phi}_j(k) \tilde{u}_j$ where for all $j \in \{1, ..., q\}$, for all $k \in K$, $\phi_j(k) = \frac{1}{2q} - \frac{1}{2r} w_j(k)$ and $\tilde{\phi}_j(k) = \frac{1}{2q} + \frac{1}{2r} w_j(k)$. For all $k \in K$, we have for all

$j \in \{1, ..., q\}$, $\phi_j(k) \geq 0$, $\tilde{\phi}_j(k) \geq 0$ and $\sum_{j=1}^q \phi_j(k) + \sum_{j=1}^q \tilde{\phi}_j(k) = 1$ therefore we have $\Phi(K) \subset K$. Since for all $j \in \{1, ..., q\}$, ϕ_j and $\tilde{\phi}_j$ belong to $C^0(K, \mathbb{R})$, we have $\Phi \in C^0(K, K)$. Therefore, by using Theorem 3.2, we obtain there exists $\hat{k} \in K$ such that $\Phi(\hat{k}) = \hat{k}$. Since

$$\begin{aligned} D_H H(\hat{x})\hat{k} &= D_H H(\hat{x})\Phi(\hat{k}) \\ &= \sum_{j=1}^q (\tfrac{1}{2q} - \tfrac{1}{2r} w_j(\hat{k})) D_H H(\hat{x}) u_j + \sum_{j=1}^q (\tfrac{1}{2q} + \tfrac{1}{2r} w_j(\hat{k})) D_H H(\hat{x}) \tilde{u}_j \\ &= -\sum_{j=1}^q w_j(\hat{k}) e_{q,j}. \end{aligned}$$

Therefore, for all $j \in \{1, ..., q\}$, $D_H h_j(\hat{x})\hat{k} = -w_j(\hat{k}) = -(\frac{1}{\alpha} h_j(\hat{x} + \alpha\hat{k}) - D_H h_j(\hat{x})\hat{k})$ which implies that $h_j(\hat{x} + \alpha\hat{k}) = 0$.

Since $\hat{k} \in K$ there exists $(a_j, \tilde{a}_j)_{1 \leq j \leq q} \in \mathbb{R}_+^{2q}$ with $\sum_{j=1}^q a_j + \sum_{j=1}^q \tilde{a}_j = 1$ such that $\hat{k} = \sum_{j=1}^q a_j u_j + \sum_{j=1}^q \tilde{a}_j \tilde{u}_j$. For consequently, we have

$$\begin{aligned} D_H f(\hat{x})\hat{k} &= \sum_{j=1}^q a_j D_H f(\hat{x}) u_j + \sum_{j=1}^q \tilde{a}_j D_H f(\hat{x}) \tilde{u}_j \\ &= \sum_{j=1}^q a_j (r - z_{0,j}) + \sum_{j=1}^q \tilde{a}_j (r - \tilde{z}_{0,j}) \text{ (from (5.1) and (5.2))} \end{aligned}$$

which implies that

$$D_H f(\hat{x})\hat{k} \geq r. \tag{5.8}$$

By the same reasoning, we have also

$$\forall i \in \{1, ..., s\}, D_H g_i(\hat{x})\hat{k} \geq r. \tag{5.9}$$

By using (5.8), (5.9), (5.5) and (5.6) with $t = \hat{k}$, we have $f(\hat{x} + \alpha\hat{k}) > f(\hat{x})$ and for all $i \in \{1, ..., s\}$, we have $g_i(\hat{x} + \alpha\hat{k}) > 0$. Since $\hat{x} + \alpha\hat{k} \in U$ verify for all $i \in \{1, ..., s\}$, $g_i(\hat{x} + \alpha\hat{k}) > 0$, for all $j \in \{1, ..., q\}$, $h_j(\hat{x} + \alpha\hat{k}) = 0$ and $f(\hat{x} + \alpha\hat{k}) > f(\hat{x})$, we have $\hat{x}$ is not a solution of $(\mathcal{NP})$. This is a contradiction. Since $0 \in C$ and C is not a neighborhood of 0, we have $0 \in \mathrm{bd}C$.

Since C is a convex of $\mathbb{R}^{1+s+q}$ and $0 \in \mathrm{bd}C$, by using Theorem 3.1 there exists $v = (\lambda_0, ..., \lambda_s, \mu_1, ..., \mu_q) \in \mathbb{R}^{1+s+q} \setminus \{0\}$ such that for all $x \in C$, $\langle v, x \rangle \leq 0$. Therefore, we have

$$\left.\begin{aligned} &\forall u \in E,\ \forall z = (z_0, ..., z_s) \in \mathbb{R}_-^{1+s} \\ &\lambda_0(D_H f(\hat{x})u + z_0) + \sum_{i=1}^s \lambda_i (D_H g_i(\hat{x})u + z_i) + \sum_{j=1}^q \mu_j D_H h_j(\hat{x})u \leq 0 \end{aligned}\right\} \tag{5.10}$$

We set for all $i \in \{s+1, ..., p\}$ $\lambda_i = 0$. Since $(\lambda_0, ..., \lambda_s, \mu_1, ..., \mu_q) \neq 0$, we have $(\lambda_0, ..., \lambda_p, \mu_1, ..., \mu_q) \neq 0$.

Let $i \in \{0, ..., s\}$, by using (5.10) with $u = 0$ and $z = -e_{s+1,i+1}$, we have $-\lambda_i \leq 0$ which implies that $\lambda_i \geq 0$. We have also for all $i \in \{1, ..., p\}$, $\lambda_i g_i(\hat{x}) = 0$. By using (5.10), with $z = 0$, we have

$\forall u \in E$, $\lambda_0 D_H f(\hat{x})u + \sum_{i=1}^s \lambda_i D_H g_i(\hat{x})u + \sum_{j=1}^q \mu_j D_H h_j(\hat{x})u \leq 0$ which implies that $\lambda_0 D_H f(\hat{x}) + \sum_{i=1}^s \lambda_i D_H g_i(\hat{x}) + \sum_{j=1}^q \mu_j D_H h_j(\hat{x}) = 0$. Since, for all $i \in \{1, ..., s\}$, $D_G g_i(\hat{x}) = D_H g_i(\hat{x})$, we have

$$\lambda_0 D_H f(\hat{x})u + \sum_{i=1}^s \lambda_i D_G g_i(\hat{x})u + \sum_{j=1}^q \mu_j D_H h_j(\hat{x})u = 0. \tag{5.11}$$

Therefore $\lambda_0 D_H f(\hat{x})u + \sum_{i=1}^p \lambda_i D_G g_i(\hat{x})u + \sum_{j=1}^q \mu_j D_H h_j(\hat{x})u = 0$. We proved (a), (b) and (c).

5.3. **Proof of (d).** We assume (i), (ii), (iii) and (iv). We proceed by contradiction by assuming that $(\lambda_0, ..., \lambda_p) = (0, ..., 0)$. Therefore, according to (5.11), $\sum_{j=1}^{q} \mu_j D_H h_j(\hat{x})u = 0$. Since (iv), we have $(\mu_1, ..., \mu_q) = 0$. For consequently, we have $(\lambda_0, ..., \lambda_p, \mu_1, ..., \mu_q) = (0, ..., 0)$ this a contradiction with (a). We proved (d).

5.4. **Proof of (e).** We assume (i), (ii), (iii), (iv) and (v). Thanks to our previous proof we know that there exist $\lambda_0, ..., \lambda_p \in \mathbb{R}_+$ and $\mu_1, ..., \mu_q \in \mathbb{R}$ which verify (a), (b), (c) and (d). We have $\lambda_0 \neq 0$, we proceed by contradiction by assuming that $\lambda_0 = 0$. Since (d) and (b), we have $(\lambda_1, ..., \lambda_s) \neq 0$. Since (v) and $(\lambda_1, ..., \lambda_s) \in \mathbb{R}_+^s \setminus \{0\}$, we have $\sum_{i=1}^{s} \lambda_i D_G g_i(\hat{x})w > 0$. By using (5.11), we have $\sum_{i=1}^{s} \lambda_i D_G g_i(\hat{x})w = 0$. This a contradiction. Since, $\lambda_0 \neq 0$, by taking for all $i \in \{0, ..., p\}$, $\lambda'_i = \frac{\lambda_i}{\lambda_0}$, we proved (e).

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